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The Discrete Time $Geo/G/1$ Queue with Negative Customers and Multiple Working Vacations

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Abstract—In **this article, we study the discrete time $Geo/G/1$ queue with negative customers and multiple working vacations. Under the RCH disciple, the arrival of a negative customer does not receive service and removes the positive customer who is being served. Using embedded Markov chain and matrix analytic approach, we derive the probability generating function (PGF) and mean of the stationary queue length at the departure epoch. From the process of the proof and the results, we also obtain the probabilities that the server is in idle, working vacation, and regular busy period, respectively. Finally, we discuss numerical results.**

Keywords—**Multiple vacation, Negative customers, Discrete time, PGF**

I. Introduction

In the past twenty years, discrete-time vacation queue model has received a rapid increase of research attention both in queuing theorists and practitioners because of its wide applicability in the study of production and inventory systems, the performance analysis of digital communication systems, and telecommunication networks including the Broadband Integrated Services Digital Network (B-ISDN), Asynchronous transfer mode (ATM), and related technologies.

Queues with negative customers have been studied over decades, and the work about negative customers without working vacation in discrete time can be found in Atencia and Moreno[1, 2], where the authors considered the single server discrete time queue with negative arrivals and various killing disciplines caused by the negative customers. And Jinting Wang and Peng Zhang[3] discussed a discrete time retrial queue with negative customers and unreliable server. Park and Yang [4] analyzed the $Geo/G/1$ queue with negative customers and disasters. However the literatures are still much fewer in the model of the $Geo/G/1$ queue with negative customers.

For discrete-time queues with vacation policies, Zhang and Tian [5] treated the discrete time $Geo/G/1$ system with variant vacations in 2001. In this system, after serving all customers the server will take a random maximum number of vacations before returning to the service. Meanwhile, the discrete time queues exactly analysis with different vacation policies can be found in Takagi [6], Tian and Alfa [7], Zhang [8], and references therein. In these papers, they all assumed that the server stop service completely during the vacations. Recently, Li [9] studied the discrete time $Geo/G/1$ working vacation queue and its application to network scheduling. But there is no negative customer to join discussion. The negative customers are the equal of all interference elements in real life. For example, in the bank queuing service process if some customers have to stop queuing due to special reasons of the individual, then we can seen these personal special reasons as negative customers; Another example, in the process of cell transfer if the interference of a virus causes information element transmission failure, then some kinds of virus can be seen as negative customers. Hence vacation queue with negative customers is relatively reasonable in real life.

On base of analysis above, in this paper we deal with the discrete-time $Geo/G/1$ queue with negative customers and multiple working vacations. Under the RCH disciple, the arrival of a negative customer does not receive service and removes the positive customer who is being served. Yet if the system is empty and the arrival of the negative customers, the negative customers will vanish and have no impact on the server. Once a positive customer arrives and the server is idle, the service of the arriving customer commences immediately. If the server completes all customers' service and finds the system becoming empty, it will leave for a vacation. During the vacation period, if there are new positive customers arriving, the server will provide service for them by their arrival order at a slower rate. At a vacation completion instant, if the server finds customers in the system, he will come back to the normal working level and take the service with the normal service time. Otherwise, it immediately proceeds for another vacation and continues in this manner until he finds positive customers in the system. If one customer is still being served when the vacation ends, we will assume that this customer will begin to receive the service as a new one.

The organization of the paper is as follows. In Section 2 we give a model description and embedded Markov chain. In Section 3 we use embedded Markov chain and matrix analytic approach and derive the probability generating function (PGF) and mean of the stationary queue length at the departure epoch. From the process of the proof and the results, we also obtain the probabilities that the server is in idle, working vacation, and regular busy period, respectively. Finally, we give a special case. In the Section 4, we discuss numerical results.

II. Model description and embedded Markov chain

The features of the discrete time $Geo/G/1$ queue with negative customers and multiple working vacations studied here are as follows:

(1) We consider a discrete-time queuing system where the time axis is segmented into a sequence of equal time intervals. It is assumed that all queuing activities (arrivals and departures) occur at the slot boundaries, and therefore they may occur at the same time. For mathematical clarity, we suppose that a potential customer's arrival occurs in (n, n^+), $n=0,1,2,\ldots$, and a potential customer's departure takes place in (n^-, n), $n=1,2,\ldots$, where $n^- = \lim_{\Delta n\to 0}(n-\Delta n)$ and $n^+ = \lim_{\Delta n\to 0}(n+\Delta n)$. This model is known as the arrival first system.

(2) Two types of customers, positive and negative, arrive according to geometrical arrival processes with the parameters p and q, respectively. Here we have to state the assumption regarding the order of these concurrent events because a positive customer's arrival and a negative customer's arrival can simultaneously occur at the same slot boundary. We also assume that the potential negative customer's arrival occurs at $t = n^+$ before the potential positive customer's arrival. Under the RCH disciple, the arrival of a negative customer does not receive service and removes the positive customer who is being served. However, if the system is empty and the negative customers arrive, the negative customers will vanish and have no impact on the server.

(3) Once a positive customer arrives and the server is idle, the service of the arriving customer commences immediately. Otherwise, the arriving customer joins the waiting line to be served. Let customers be served at slot boundaries (n, n^+), $n = 0,1,2,\ldots$. Further, the service of a customer takes an integer number of slots, which implies that customers also leave the system at slot boundaries, and so suppose departure the sever in $((n+1)^-, (n+1))$, $n = 0,1,2,\ldots$. Every customer's service time S_b is independent and obeys the general discrete distribution during regular busy period. Its general distribution, probability generating function (PGF), and mean are $g_k^{(1)}, k = 1,2,\ldots$, $G_b(z)$, and $E(S_b)$, respectively.

(4) If the server completes all customers' services and finds the system being empty, it will leave for a vacation. During the vacation period if there are new customers arriving, the server provides service for them by their arrival order at a slower rate. Service time S_v of every customer is independent and obeys the general discrete distribution in working vacations. Its general distribution, probability generating function (PGF), and mean are $g_k^{(2)}$, $k = 1,2,\ldots$, $G_v(z)$, and $E(S_v)$, respectively.

(5) At a vacation completion instant, if the server finds customers in the system, it will come back to the normal working level and take the service with the normal service time. On the other hand, if it finds no customer, it immediately proceeds for another vacation and continues in this manner until it finds customers in the system. If one customer is still being served when the vacation ends, we will assume that this customer will begin to receive the service as a new one.

(6) There is only a server in the system, and service rule is first come first service, and the server can serve only one customer at a time. The vacation time V obeys geometrical distribution with the parameter θ

$$P(V = k) = \theta\bar{\theta}^{k-1}, k = 1,2,\ldots, V(z) = \frac{\theta z}{1-\bar{\theta} z}, E(V) = \frac{\theta}{(1-\theta)^2}.$$

We assume that inter arrival times, the normal service time, and working vacation time and normal vacation time are mutually independent. Thereinafter, for any real number $x \in [0,1]$, we denote $\bar{x} = 1 - x$.

Let L_n be the number of the customers at the instant of the nth service completion or customer departure, and J_n the state of server both at a slot division point t^+, $t = 0,1,2,\ldots$. For the working vacation model, any service completion may occur during a service period or a working vacation period. Thus, we define a process

$$X_n = (L_n, J_n), n \geq 1,$$

where J_n symbolizes the system state at the nth customer departure epoch ($J_n = 0$ if the system stays in a working vacation period, or $J_n = 1$ if the system stays in a busy service period). Then $\{X_n, n \geq 1\}$ is a two-dimensional embedded Markov chain of our queue system and its state space is $\Omega = (0,0) \cup \{(k, j), k \geq 1, j = 0,1\}$.

The server will stay in the vacation period if there are no customers in the system.

Let a_j, $j \geq 0$ be the probability that there are j customer arrivals during the service time S_b. Let the service time S_b of a positive customer be equal to k in the regular busy period. Then we get the following two cases:

(i) The service of a positive customer ends;

(ii) The service of a positive customer doesn't end, but he be removed by a negative customer.

Then we can get that

$$a_j = \sum_{k=\max(1,j)}^{\infty} g_k^{(1)} \binom{k}{j} p^j (1-p)^{k-j} (1-q)^k + \sum_{k=\max(1,j)}^{\infty} g_k^{(1)} \binom{k}{j} p^j (1-p)^{k-j} (1-q)^k q$$

$$= (1+q) \sum_{k=\max(1,j)}^{\infty} g_k^{(1)} \binom{k}{j} p^j (1-p)^{k-j} (1-q)^k, \qquad j \geq 0.$$

And the sum of all possible arrival probability in the service time S_b is

$$\sum_{j=0}^{\infty} a_j = \sum_{j=0}^{\infty} (1+q) \sum_{k=\max(1,j)}^{\infty} g_k^{(1)} \binom{k}{j} p^j (1-p)^{k-j} (1-q)^k$$

$$= (1+q) G_b(\bar{q}).$$

Since the sum of all possible arrival probability during the service time S_b is equal to 1, one has $G_b(\bar{q}) = \frac{1}{1+q}$.

Hence their PGF and means are

$$A(z) = \sum_{j=0}^{\infty} a_j z^j = (1+q) \sum_{j=0}^{\infty} \sum_{k=j}^{\infty} g_k^1 \binom{k}{j} (pz)^j (1-p)^{k-j} (1-q)^k = (1+q) G_b(\bar{p}\bar{q} + p\bar{q}z)$$

,

$$A'(z)\Big|_{z=1} = \frac{(1+q)\bar{q}p}{\mu_b} = \rho, \qquad A''(z)\Big|_{z=1} = (1+q)(\bar{q}p)^2 G_b''(\bar{q}).$$

Similarly, let b_j ($j = 0,1,\ldots$) be the probability that $V > S_v$ and j customers arrive during the service time S_v and v_j ($j = 0,1,\ldots$) be the probability that $V \leq S_v$ and j customers arrive in the vacation time V. Therefore, we can denote

$$b_j = (1+q) \sum_{k=\max(1,j)}^{\infty} g_k^{(2)} \binom{k}{j} p^j (1-p)^{k-j} (1-q)^k (1-\theta)^k,$$

$$v_j = (1+q) \sum_{k=\max(1,j)}^{\infty} \sum_{n=k}^{\infty} g_n^{(2)} (1-q)^n \binom{k}{j} p^j (1-p)^{k-j} (1-\theta)^{k-1} \theta.$$

And so, we have that

$$\sum_{j=0}^{\infty} b_j = (1+q) G_v(\bar{q}\bar{\theta}), \sum_{j=0}^{\infty} v_j = (1+q)\left(G_v(\bar{q}) - G_v(\bar{q}\bar{\theta})\right).$$

Since the sum of all possible arrival probabilities in the service time S_v equals to 1, we get that $G_v\left(\bar{q}\right)=\frac{1}{1+q}$. Thus, we have the PGF and means of them in the following:

$$B(z)=\sum_{j=0}^{\infty}b_j z^j=\sum_{j=0}^{\infty}(1+q)\sum_{k=\max(1,j)}^{\infty}g_k^{(2)}\binom{k}{j}(pz)^j(1-p)^{k-j}(1-q)^k(1-\theta)^k$$
$$=(1+q)G_v\left(\bar{\theta}\bar{q}\left(\bar{p}+pz\right)\right),$$

$$B'(z)\big|_{z=1}=(1+q)\bar{\theta}\bar{q}pE(G_v)=\frac{(1+q)\bar{\theta}\bar{q}p}{\mu_v}=\beta,$$

$$B''(z)\big|_{z=1}=(1+q)\left(\bar{\theta}\bar{q}p\right)^2G_v''\left(\bar{\theta}\bar{q}\right)$$

$$V(z)=\sum_{j=0}^{\infty}v_j z^j$$
$$=\sum_{j=0}^{\infty}(1+q)\sum_{k=\max(1,j)}^{\infty}\sum_{n=k}^{\infty}g_n^{(2)}(1-q)^n\binom{k}{j}(pz)^j(1-p)^{k-j}(1-\theta)^{k-1}\theta$$
$$=\frac{(1+q)\theta}{1-\bar{\theta}\left(\bar{p}+pz\right)}\sum_{n=k}^{\infty}g_n^{(2)}(1-q)^n\left[1-\left(\bar{\theta}\left(\bar{p}+pz\right)\right)^n\right]$$
$$=\frac{(1+q)\theta}{1-\bar{\theta}\left(\bar{p}+pz\right)}\left[G_v\left(\bar{q}\right)-G_v\left(\bar{q}\bar{\theta}\left(\bar{p}+pz\right)\right)\right]$$
$$=\frac{\theta}{1-\bar{\theta}\left(\bar{p}+pz\right)}\left[(1+q)G_v\left(\bar{q}\right)-B(z)\right]$$
$$=\frac{\theta}{1-\bar{\theta}\left(\bar{p}+pz\right)}\left[1-B(z)\right],$$

$$V'(z)\big|_{z=1}=\frac{p\bar{\theta}}{\theta}\left[1-(1+q)G_v\left(\bar{\theta}\bar{q}\right)\right]-\beta,$$

$$V''(z)\big|_{z=1}=-B''(1)-\frac{2\bar{\theta}pB'(1)}{\theta}+\frac{2\left(\bar{\theta}p\right)^2\left[1-B(1)\right]}{\theta^2}.$$

Let $c_j=\sum_{k=0}^{j}v_k a_{j-k},\ j\ge 0$. Then c_j represents the probability that $V\le S_v$ and j customers arrive in the vacation time V and service time S_b, and we have

$$\sum_{j=0}^{\infty}c_j=\sum_{j=0}^{\infty}\sum_{k=0}^{j}v_k a_{j-k}=\sum_{k=0}^{\infty}v_k\sum_{j=k}^{\infty}a_{j-k}=1-(1+q)G_v\left(\bar{q}\bar{\theta}\right),$$
$$C(z)=\sum_{j=0}^{\infty}c_j z^j=\sum_{j=0}^{\infty}\sum_{k=0}^{j}v_k a_{j-k}z^j=\sum_{k=0}^{\infty}v_k z^k\sum_{j=k}^{\infty}a_{j-k}z^{j-k}$$
$$=A(z)V(z),$$
$$C'(z)\big|_{z=1}=A'(1)V(1)+A(1)V'(1)$$
$$=\left(\rho+\frac{p\bar{\theta}}{\theta}\right)\left[1-(1+q)G_v\left(\bar{\theta}\bar{q}\right)\right]-\beta,$$
$$C''(z)\big|_{z=1}=A''(1)V(1)+2A'(1)V'(1)+A(1)V''(1).$$

Now, we consider the one-step transition probabilities of $(L_n,J_n),n\ge 1$, and get the following results.

If $X_n=(m,1),\ m\ge 2$, then the remaining customers are those $m-1$ customers and new arriving ones during the service time S_b at the next departure epoch, and thus $X_{n+1}=(m-1+j,1)$ with probability $a_j,\ m\ge 2,\ j\ge 0$.

If $X_n=(1,1)$, then two cases happen. On the one hand, the remaining customers are those that have arrived during the normal service time S_b at the next departure epoch. on the other hand, no customers arrive in the normal service time S_b and the system becomes empty. Thus, $X_{n+1}=(j,1)$ with probability $a_j,\ j\ge 1$ and $X_{n+1}=(0,0)$ with probability a_0.

If $X_n=(m,0),\ m\ge 2$, then there also are two cases at the next departure epoch.

Case1: if $V>S_v$, after this departure, the system stay in the vacation period and the remaining customers are those $m-1$ customers and new arriving customers in the service time S_v. Hence we have $X_{n+1}=(m-1+j,0)$ with probability $b_j,\ j\ge 0$.

Case 2: if $V\le S_v$, next departure happens in the normal service period and the remaining customers are those $m-1$ customers and new arriving customers during the vacation time V and the service time S_b. By the definition of b_j and c_j, we have $X_{n+1}=(m-1+j,1)$ with probability $v_j,\ j\ge 0$.

If $X_n=(m,0)$, $m=0,1$, then by the similar analysis, we get that $X_{n+1}=(j,0)$ with probability b_j, $j\ge 1$, $X_{n+1}=(j,1)$ with probability c_j, $j\ge 1$, and $X_{n+1}=(0,0)$ with probability b_0+c_0.

Using the lexicographical sequence for the states, the transition probability matrix of (L_n,J_n) can be written as the Block-Jacobi matrix

$$\tilde{P}=\begin{bmatrix} B_0 & B_1 & B_2 & B_3 & \cdots \\ C_0 & A_1 & A_2 & A_3 & \cdots \\ & A_0 & A_1 & A_2 & \cdots \\ & & A_0 & A_1 & \cdots \\ & & & \vdots & \vdots \end{bmatrix},$$

where $B_0=b_0+c_0$; $B_i=(b_i,c_i)$, $i\ge 1$; $C_0=(b_0+c_0,a_0)^T$; $A_i=\begin{bmatrix} b_i & c_i \\ & a_i\end{bmatrix}, i\ge 0$.

And the matrix $\tilde{P}$ satisfies the following conditions

$$B_0+\sum_{i=1}^{\infty}B_i e=1;\ C_0+\sum_{i=1}^{\infty}A_i e=e;\ \sum_{i=0}^{\infty}A_i e=e,\ e=(1,1).$$

Next, we give out a lemma which is crucial to prove the theorem in the sequel.

Lemma 1. If $\rho=\frac{(1+q)\bar{q}p}{\mu_b}<1$ and $\theta>0$, then the matrix equation $G=\sum_{i=0}^{\infty}A_i G^i$ has the minimal nonnegative solution

$$G=\begin{bmatrix}\gamma & 1-\gamma \\ & 1\end{bmatrix},$$


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where γ is the unique root in the range $0<z<1$ of the equation $z=(1+q)G_v\left(\bar{\theta}\bar{q}\left(\bar{p}+pz\right)\right)$.

Proof. Since all A_i's are upper triangular, we can assume that the minimal nonnegative solution G has the same structure as

$$G=\begin{bmatrix} r_{11} & r_{12} \\ 0 & r_{22} \end{bmatrix}.$$

If $i\geq 1$, we have

$$G^i=\begin{bmatrix} r_{11}^i & r_{12}\sum_{j=0}^{i-1} r_{11}^j r_{22}^{i-1-j} \\ 0 & r_{22}^i \end{bmatrix}.$$

Substituting G, G^i, and A_i into the matrix equation $G=\sum_{i=0}^{\infty}A_iG^i$, we obtain

$$r_{11}=\sum_{i=0}^{\infty}b_i r_{11}^i=(1+q)G_V\left(\bar{\theta}\bar{q}\left(\bar{p}+pr_{11}\right)\right),$$

$$r_{12}=\sum_{i=0}^{\infty}b_i r_{12}\sum_{j=0}^{i-1}r_{11}^j r_{22}^{i-1-j}+\sum_{i=0}^{\infty}c_i r_{22}^i,$$

$$r_{22}=\sum_{i=0}^{\infty}a_i r_{22}^i=(1+q)G_b\left(\bar{q}\left(\bar{p}+pr_{22}\right)\right).$$

Firstly, we consider the equation

$$z=(1+q)G_b\left(\bar{q}\left(\bar{p}+pz\right)\right),$$

and let $f(z)=(1+q)G_b\left(\bar{q}\left(\bar{p}+pz\right)\right)$.

Then

$$f(0)=(1+q)G_b\left(\bar{p}\bar{q}\right)=\frac{G_b\left(\bar{p}\bar{q}\right)}{G_b\left(\bar{q}\right)},$$

$$f(1)=(1+q)G_b\left(\bar{q}\right)=1,$$

and so $0<f(0)<f(1)$.

Moreover, for any z, $0<z<1$, we get

$$f'(z)=(1+q)p\bar{q}G_b'\left(\bar{q}\left(\bar{p}+pz\right)\right)>0,$$

$$f''(z)=(1+q)\left(p\bar{q}\right)^2G_b''\left(\bar{q}\left(\bar{p}+pz\right)\right)>0.$$

Since $f'(1)=(1+q)p\bar{q}G_b'\left(\bar{q}\right)=\rho<1$, it follows that the equation $z=f(z)$ has the minimal nonnegative root $z=1$, and thus $r_{22}=1$.

Secondly, let $g(z)=z-(1+q)G_v\left(\bar{\theta}\bar{q}\left(\bar{p}+pz\right)\right)$. Then we can get that

$$g(0)=-(1+q)G_v\left(\bar{\theta}\bar{q}\bar{p}\right)<0,$$

$$g(1)=1-(1+q)G_v\left(\bar{\theta}\bar{q}\right)=1-\frac{G_v\left(\bar{\theta}\bar{q}\right)}{G_v\left(\bar{q}\right)}>0.$$

Further, since

$$y=(1+q)G_v\left(\bar{\theta}\bar{q}\left(\bar{p}+pz\right)\right)$$

is an increasing function for $0<z<1$. Thus, the equation

$$z=(1+q)G_v\left(\bar{\theta}\bar{q}\left(\bar{p}+pz\right)\right)$$

has a unique root γ, $0<\gamma<1$, and so $r_{11}=\gamma$.

Finally, taking $r_{11}=\gamma$ and $r_{22}=1$ into the second equation, we easily have that $r_{12}=1-\gamma$.

In view of the structure of matrix G, G is a stochastic matrix.

III. Queue length and main results

Let (L,J) be the stationary limit of $X_n=(L_n,J_n)$. Then the stationary distribution can be defined by

$$\pi_{kj}=P(L=k,J=j)=\lim_{n\to\infty}P(L_n=k,J_n=j),\quad (k,j)\in\Omega;$$

$$\pi=(\pi_0,\pi_1,\pi_2,\ldots),\ \pi_0=\pi_{00},\ \pi_k=(\pi_{k0},\pi_{k1}).$$

Since the evolution of the chain is governed by the one-step transition probabilities given by matrix $\tilde{P}$, the Kolmogorov equations $\pi\tilde{P}=\pi$ together with the normalization condition for the stationary distributions are given by

$$\pi_0=\pi_0\left(b_0+c_0\right)+\pi_1C_0=\pi_0\left(b_0+c_0\right)+\pi_{10}\left(b_0+c_0\right)+\pi_{11}a_0, \quad (1)$$

$$\pi_k=\pi_0B_k+\sum_{j=1}^{k+1}\pi_jA_{k+1-j},\ k\geq 1, \quad (2)$$

$$\pi_0+\sum_{k=1}^{\infty}\pi_k e=1. \quad (3)$$

From Eq. (2), we can obtain the row vector generating function

$$\begin{aligned}\Phi(z)&=\sum_{k=1}^{\infty}\pi_k z^k=\pi_0\sum_{k=1}^{\infty}B_kz^k+\sum_{k=1}^{\infty}z^k\sum_{j=1}^{k+1}\pi_jA_{k+1-j}\\ &=\pi_0\sum_{k=1}^{\infty}B_kz^k+\frac{1}{z}\Phi(z)A^*(z)-\pi_1A_0,\end{aligned} \quad (4)$$

where $A^*(z)=\sum_{k=0}^{\infty}A_kz^k=\begin{bmatrix} B(z) & C(z) \\ 0 & A(z)\end{bmatrix}$.

Furthermore,

$$\begin{aligned}\Phi(z)&=z\left(\pi_0\left(B(z)-b_0,C(z)-c_0\right)-\left(\pi_{10}b_0,\pi_{10}c_0+\pi_{11}a_0\right)\right)\begin{bmatrix} z-B(z) & -C(z) \\ 0 & z-A(z)\end{bmatrix}^{-1}\\ &=z\left(\pi_0B(z)-b_0\left(\pi_0+\pi_{10}\right),\pi_0\left(C(z)-1\right)+b_0\left(\pi_0+\pi_{10}\right)\right)\begin{bmatrix} \frac{1}{(z-B(z))} & \frac{C(z)}{(z-A(z))(z-B(z))} \\ 0 & \frac{1}{(z-A(z))}\end{bmatrix}\\ &=z\left(\frac{\left[\pi_0B(z)-b_0\left(\pi_0+\pi_{10}\right)\right]}{(z-B(z))}+\frac{C(z)\left[\pi_0B(z)-b_0\left(\pi_0+\pi_{10}\right)\right]}{(z-A(z))(z-B(z))}+\frac{\pi_0\left(C(z)-1\right)+b_0\left(\pi_0+\pi_{10}\right)}{(z-A(z))}\right).\end{aligned} \quad (5)$$

Now, we give out the main theorem and its proof.

Theorem 1. If $\rho=\frac{(1+q)\bar{q}p}{\mu_b}<1$, then the stationary queue length L at the departure epoch has the following generating function:

$$L(z)=\frac{A(z)(z-1)\left(z-B(z)\right)+z(\gamma-z)\left(A(z)-B(z)-C(z)\right)}{\left(z-A(z)\right)\left(z-B(z)\right)}\pi_0, \quad (6)$$

where

$$\pi_0=\frac{(1-\rho)\left(1-(1+q)G_v\left(\bar{q}\bar{\theta}\right)\right)}{1+(1-\gamma)\frac{p\bar{\theta}}{\theta}-\left(1+(1-\gamma)\left(\rho+\frac{p\bar{\theta}}{\theta}\right)\right)(1+q)G_v\left(\bar{q}\bar{\theta}\right)}$$

$$A(z)=(1+q)G_b\left(\bar{p}\bar{q}+p\bar{q}z\right),\ B(z)=(1+q)G_V\left(\bar{\theta}\bar{q}\left(\bar{p}+pz\right)\right),$$

$$C(z)=A(z)V(z),V(z)=\frac{\theta}{1-\bar{\theta}\left(\bar{p}+pz\right)}\left[1-B(z)\right]. \qquad (7)$$

Proof. The PGF $L(z)$ of the stationary queue length at the departure epoch can be derived

$$L(z)=\pi_0+\Phi(z)e$$
$$=\pi_0+\frac{z\left[\pi_0B(z)-b_0\left(\pi_0+\pi_{10}\right)\right]}{\left(z-B(z)\right)}+$$
$$\frac{zC(z)\left[\pi_0B(z)-b_0\left(\pi_0+\pi_{10}\right)\right]}{\left(z-A(z)\right)\left(z-B(z)\right)}+\frac{z\left[\pi_0\left(C(z)-1\right)+b_0\left(\pi_0+\pi_{10}\right)\right]}{\left(z-A(z)\right)}$$

$$=\frac{\pi_0\left[-zA(z)+A(z)B(z)+z^2B(z)-zA(z)B(z)+z^2C(z)\right]}{\left(z-A(z)\right)\left(z-B(z)\right)}$$
$$+\frac{+zb_0\left(\pi_0+\pi_{10}\right)\left(A(z)-B(z)-C(z)\right)}{\left(z-A(z)\right)\left(z-B(z)\right)}. \qquad (8)$$

By Lemma 1, we know that $\gamma\,(0<\gamma<1)$ is the unique root of the equation

$$z=(1+q)G_v\left(\bar{\theta}\bar{q}\left(\bar{p}+pz\right)\right)=B(z).$$

Hence the denominator of Eq. (8) equals 0 if $z=\gamma$, and so the numerator of Eq. (8) also equals 0 if $z=\gamma$. Substituting $z=\gamma$ into the numerator of
Eq. (8), we can get

$$\gamma\pi_0=b_0\left(\pi_0+\pi_{10}\right)\Rightarrow\pi_{10}=\frac{\left(\gamma+b_0\right)}{b_0}\pi_0 \qquad (9)$$

Taking Eq. (9) into Eq. (8), the Eq. (8) can be converted into Eq. (6). Using Eq. (8) and the normalizing condition $L(1)=1$, Eq. (7) follows.

Corollary 1. Assume that the discrete time $Geo/G/1$ queue with negative customers and multiple working vacations is in the steady state. If $\rho=\frac{(1+q)\bar{q}p}{\mu_b}<1$, then we have the following results:

(1) The probability that the server is idle is

$$\pi_0=\frac{(1-\rho)\left(1-(1+q)G_v\left(\bar{q}\bar{\theta}\right)\right)}{1+(1-\gamma)\frac{p\bar{\theta}}{\theta}-\left(1+(1-\gamma)\left(\rho+\frac{p\bar{\theta}}{\theta}\right)\right)(1+q)G_v\left(\bar{q}\bar{\theta}\right)}.$$

(2) The probability that the server is in working vacation is

$$\Phi(z)\big|_{z=1}e_1=\frac{(1-\rho)\left[(1+q)G_v\left(\bar{q}\bar{\theta}\right)-\gamma\right]}{1+(1-\gamma)\frac{p\bar{\theta}}{\theta}-\left(1+(1-\gamma)\left(\rho+\frac{p\bar{\theta}}{\theta}\right)\right)(1+q)G_v\left(\bar{q}\bar{\theta}\right)}.$$

(3) The probability that the server is in vacation is

$$P(J=0)=\pi_0+\Phi(z)\big|_{z=1}e_1$$
$$=\frac{(1-\rho)(1-\gamma)}{1+(1-\gamma)\frac{p\bar{\theta}}{\theta}-\left(1+(1-\gamma)\left(\rho+\frac{p\bar{\theta}}{\theta}\right)\right)(1+q)G_v\left(\bar{q}\bar{\theta}\right)}.$$

(4) The probability that the server is in regular busy period is

$$P(J=1)=\Phi(z)\big|_{z=1}e_2$$
$$=\frac{\gamma+(1-\gamma)\left(\rho+\frac{p\bar{\theta}}{\theta}\right)-\left(1+(1-\gamma)\left(\rho+\frac{p\bar{\theta}}{\theta}\right)\right)(1+q)G_v\left(\bar{q}\bar{\theta}\right)}{1-\frac{p\bar{\theta}}{\theta}-\left(1+(1-\gamma)\left(\rho+\frac{p\bar{\theta}}{\theta}\right)\right)(1+q)G_v\left(\bar{q}\bar{\theta}\right)}.$$

Where $e_1=(1,0)^T$, $e_2=(0,1)^T$.

Corollary 2. Under hypothesis of Corollary 1, we have the mean of the stationary queue length L :

$$E(L)=\frac{1}{1+(1-\gamma)\frac{p\bar{\theta}}{\theta}-\left(1+(1-\gamma)\left(\rho+\frac{p\bar{\theta}}{\theta}\right)\right)(1+q)G_v\left(\bar{q}\bar{\theta}\right)}$$
$$\left\{1+\rho(\gamma-2)+\beta\left[-1+(\gamma-1)+(\gamma-1)\left(\rho+\frac{\bar{\theta}p}{\theta}\right)\right]\right.$$
$$+(1+q)^2\left(\bar{q}p\right)^2(\gamma-1)G_v\left(\bar{q}\bar{\theta}\right)G_b''\left(\bar{q}\right)+\left[1-(1+q)G_v\left(\bar{q}\bar{\theta}\right)\right]$$
$$\left.\left[\rho-(\gamma-2)\left(\rho+\frac{p\bar{\theta}}{\theta}\right)-\rho\left(\rho+\frac{p\bar{\theta}}{\theta}\right)(\gamma-1)-(\gamma-1)\frac{\left(\bar{\theta}p\right)^2}{\theta^2}\right]\right\}.$$

In this place, we will list a special case of our results which is consistent with Li [9].

Case: If we consider the case $q=0$, the model converts into the discrete time $Geo/G/1$ queue with multiple working vacations, and the stationary queue length L :

$$L(z)=\frac{A(z)(z-1)\left(z-B(z)\right)+z(\gamma-z)\left(A(z)-B(z)-C(z)\right)}{\left(z-A(z)\right)\left(z-B(z)\right)}\pi_0,$$

where

$$\pi_0=\frac{(1-\rho)\left(1-G_v\left(\bar{\theta}\right)\right)}{1+(1-\gamma)\frac{p\bar{\theta}}{\theta}-\left(1+(1-\gamma)\left(\frac{p}{\mu_b}+\frac{p\bar{\theta}}{\theta}\right)\right)G_v\left(\bar{\theta}\right)},$$

$$V(z)=\frac{\theta}{1-\bar{\theta}\left(\bar{p}+pz\right)}\left[1-B(z)\right],$$

$$A(z)=G_b\left(\bar{p}+pz\right),\qquad B(z)=G_V\left(\bar{\theta}\left(\bar{p}+pz\right)\right),$$

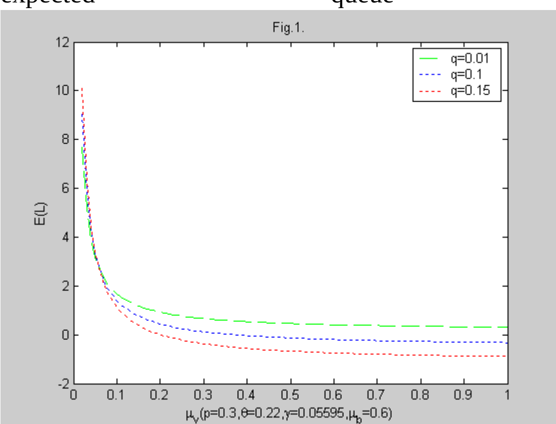
$$C(z)=A(z)V(z).$$

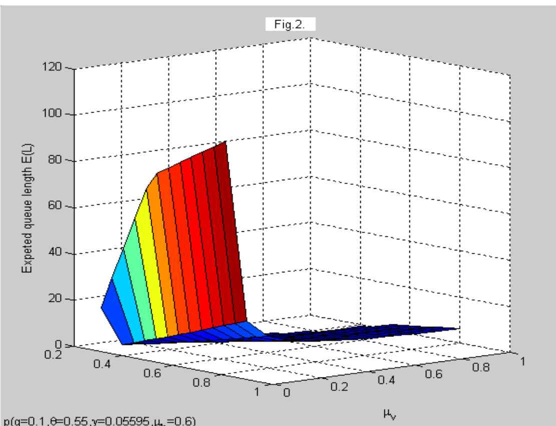
IV. Numerical results

In this section, we present some numerical results to study the effect of the varying parameters on the main performance

characteristics of our system. The values of the parameters are chosen so as to satisfy

$\rho = \frac{(1+q)\bar{q}p}{\mu_b} < 1$. And in the analysis above, we obtain the expected queue length in the steady state. Hence we will present numerical results to explain that our model is reasonable by the expected queue length.





According to the expression of the expected queue length $E(L)$, we show the effect of the service rate of working vacation μ_v on the queue length. Figure 1 is two dimension contrast figures of the expected queue length when $q = 0.01$, , $q = 0.1$, and $q = 0.15$. Since

$\rho = \frac{(1+q)\bar{q}p}{\mu_b} < 1$, we get that the effective image is the above zero. By the image in Figure 1, the number of the customers in steady state decreases along with the increase of μ_v. And we also find that the number of the customers in steady state also decreases along with the increase of the negative customer's arrival rate q when the service rate of working vacation μ_v is constant. In other words, if the parameters $p, q,$ and μ_b are fixed in this model, then negative customers arrival rate q is more higher, the number of the customers decreases more faster under the steady state, and while the system is more likely to empty. By Figure 2, we find that the number of the customers in steady is increasing when the arrival rate of the positive customer p is increasing process, which is in accordance with the queuing system in real life.

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